Cambridge Assessment Admissions Testing

Sixth Term Examination Paper [STEP]

Mathematics 1 [9465]

2019

Examiner's Report

Hints and Solutions

Mark Scheme

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Contents

STEP MATHEMATICS 1

2019

Examiner's Report

SI (9465) 2019 Report

General Comments

In order to get the fullest picture, this document should be read in conjunction with the question paper, the marking scheme and (for comments on the underlying purpose and motivation for finding the right solution-approaches to questions) the *Hints and Solutions* document; all of which are available from the STEP and Cambridge Examinations Board websites.

The purpose of the STEPs is to learn what students are able to achieve mathematically when applying the knowledge, skills and techniques that they have learned within their standard A-level (or equivalent) courses … but seldom within the usual range of familiar settings. STEP questions require candidates to work at an extended piece of mathematics, often with the minimum of specific guidance, and to make the necessary connections. This requires a very different mind-set to that which is sufficient for success at A-level, and the requisite skills tend only to develop with prolonged and determined practice at such longer questions for several months beforehand.

One of the most crucial features of the STEPs is that the routine technical and manipulative skills are almost taken for granted; it is necessary for candidates to produce them with both speed and accuracy so that the maximum amount of time can be spent in thinking their way through the problem and the various hurdles and obstacles that have been set before them. Most STEP questions begin by asking the solver to do something relatively routine or familiar before letting them loose on the real problem. Almost always, such an opening has not been put there to allow one to pick up a few easy marks, but rather to point the solver in the right direction for what follows. Very often, the opening result or technique will need to be used, adapted or extended in the later parts of the question, with the demands increasing the further on that one goes. So it is that a candidate should never think that they are simply required to 'go through the motions' but must expect, sooner or later, to be required to show either genuine skill or real insight in order to make a reasonably complete effort. The more successful candidates are the ones who manage to figure out how to move on from the given starting-point.

Finally, reading through a finished solution is often misleading – even unhelpful – unless you have attempted the problem for yourself. This is because the thinking has been done for you. When you read through the report and look at the solutions (either in the mark-scheme or the *Hints & Solutions* booklet), try to figure out how you could have arrived at the solution, learn from your mistakes and pick up as many tips as you can whilst working through past paper questions.

This year's paper produced the usual sorts of outcomes, with far too many candidates wasting valuable time by attempting more than six questions, and with many of these candidates picking up 0-4 marks on several 'false starts' which petered out the moment some understanding was required.

Around one candidate in eight failed to hit the 30 mark overall, though this is an improvement on last year. Most candidates were able to produce good attempts at two or more questions. At the top end of the scale, around a hundred candidates scored 100 or more out of 120, with four hitting the maximum of 120 and many others not far behind.

The paper is constructed so that question 1 is very approachable indeed, the intention being to get everyone started with some measure of success; unsurprisingly, Q1 was the most popular question of all, although under two-thirds of the entry attempted it this year, and it also turned out to be the most successful question on the paper with a mean score of about 12 out of 20.

In order of popularity, Q1 was followed by Qs.3, 4 and 2. Indeed, it was the pure maths questions in Section A that attracted the majority of attention from candidates, with the applied questions combined scoring fewer 'hits' than any one of the first four questions on its own. Though slightly more popular than the applied questions, the least successful question of all was Q5, on vectors. This question was attempted by almost 750 candidates, but 70% of these scored no more than 2 marks, leaving it with a mean score of just over 3 out of 20. Q9 (a statics question) was found only marginally more appetising, with a mean score of almost 3½ out of 20.

In general, it was found that explanations were poorly supplied, with many candidates happy to overlook completely any requests for such details.

Question 1

As intended, this was the most popular question on the paper and the one that elicited the highest average score. The set-up is a familiar one for A-level, though the "coefficients" involved are trigonometric throughout; it was this side of things that provided the only real degree of difficulty to the question.

Given that there was no requirement for candidates to justify the natures of the two extrema involved in the question, the issue was how accurately candidates could manage to deploy the necessary trig. identities at the appropriate points. Interestingly, it was clear that those who made tough going of the working were the ones who handled the three *reciprocal* trig. functions – cosec, sec and cot – and their derivatives less confidently. For instance, there are those who immediately convert any trig. situation exclusively into sines and cosines … this generally works perfectly well for A-level questions. Here, it simply turned out to be a hindrance as the resulting terms would contain rational functions which required heavier-duty methods for dealing with them; and many such candidates got into a muddle somewhere along the way, especially with the square of the distance XY in part (ii), which should have been turned into a perfect square reasonably swiftly.

In order to get the given result for k towards the end of (ii), a number of candidates resorted to verification, though this worked reasonably well provided they didn't somehow confuse α s with θ s. Those who wrapped up the required final answer correctly were those who spotted the *difference-of-two-squares* factorisation embedded in there *and* spotted that $(c + s)^2 = 1 + 2sc$ at the right moment (using the usual abbreviations for cos and sin).

Question 2

Given that there has been a question of a similar nature to this on several of the STEP I's of recent years, the demands of these sorts of coordinate geometry questions have become relatively routine. Attracting the interest of 60% of the candidature, this question drew the second highest mean score overall, just over half-marks.

The first major stumbling block in this question was a lack of technical precision when taking square roots of equations. If done using parametric differentiation this was not needed, but too many students rearranged $x = 3t^2$ to form $t = \sqrt{\frac{x}{3}}$. Many candidates also assumed that if $3p^2 = 3t^2$ then p must equal t. Candidates should also be aware that not all letters are equivalent. In the first part of this question, t was a variable but p was a fixed value. Many candidates wrote $\frac{dx}{dp}$ showing a fundamental misunderstanding, although it led algebraically to the correct result in this instance.

Lots of candidates did not read the question carefully. In the second paragraph it was required that the point of intersection of the tangents be found in general. Many students who clearly could have done this conflated it with the constraint that the tangents had to be perpendicular, which was a separate question.

Finding the parameterised form of the locus of intersections of perpendicular tangents required some judicious algebra and use of $pq = -1$. This was generally done quite well, although many candidates were not aware of the difference of two cubes which made the algebra much nastier. It is often worth neatening up algebraic expressions before continuing to work with them.
When this curve was intersected with the original curve it became clear that many candidates were not clear which x coordinate was related to which point. Correct simultaneous equations usually led to a disguised cubic. Candidates seemed quite good at spotting one solution and factorising to find the remaining ones. However, not all candidates explained why one solution was not possible.

The final sketch was rarely done well. The semi-cubical parabola was meant to be unfamiliar, but most students could not piece together the information to realise that it must have a cusp. Overall, most candidates were able to engage and make progress with this question, albeit with several technical errors.

Question 3

This question was clearly found an attractive proposition, drawing almost 2000 ‘hits’, partly due to the clearly directed beginning and the immediate likelihood that the second integral could also be attacked with an exactly similar tactic: replacing the factor of $(1 - \sin x)$ given for the first integral by $(1 - \sec x)$ for the second … which, as it happens, works as well as any other method. As a result, the question was answered relatively well by those who could take advantage of the starting prompt. However, many candidates failed to make much of an impression, to the extent that a fifth of all takers scored only 0, 1 or 2 marks on it.

The main issue is that the question requires a considerable degree of dexterity in one’s approach, switching between the various integration techniques without any further signposting: “recognition” (a.k.a. “reverse chain rule”) integration was the most direct but required the clearest grasp of the various trigonometric relationships that could be used, but “substitution” and “by parts” were also necessary at times, depending upon how one split up the integrand within one’s working. Those who were most successful used the method introduced by the first part to tackle the second, but many candidates succeeded with other, sometimes considerably longer methods. Most candidates could remember integrals of simple trigonometric functions, e.g. $\sec^2 x$. Trigonometric identities were generally applied accurately, though sometimes over-zealously; for example simplifying $1 - \sin^2 x$ by a double angle formula rather than to $\cos^2 x$. The most successful candidates made judicious use of these known formulae to produce an integrand which they could integrate directly, while those who appeared less discerning in their choice of identities cycled through many expressions which were difficult to integrate. The integral of $\sec^4 x$ was the most difficult in this respect, though plenty of candidates were successful in their handling of it.

Question 4

This question was also attempted by many candidates, being the last of the "big four" early questions, and – along with questions 1 and 2 – one of the only questions for which scores exceeded 10/20 on average.

Most students did part (i) well, although it was quite common to see negative or non-integer values of m and n included in the answer, despite the question's clear wording. It was also unfortunate that many candidates went about it in the longest way imaginable, squaring $m + n\sqrt{2}$ and then comparing the result with the intended answer … *then* solving for m and n from a pair of simultaneous equations (one linear and one quadratic) when the small numbers involved, and the fact that the question stated that they would be integers, required a careful evaluation of the situation.

In part (ii), most students correctly got three equations for p, q and s by expanding the factorisation given and comparing coefficients. However, relatively few gave a clear justification that such a factorisation must exist; there were some wordy but vague attempts at this, with the logic commonly being reversed. Some candidates found it difficult to manipulate the simultaneous non-linear equations to obtain useful expressions, but there were also many very good derivations of the desired equation, following several different routes.

Most students, including those who had been unable to derive it, reduced the given equation to a cubic equation for s^2 and solved this with no problems. A good number simply spotted the roots with no working shown. In using the value $s = \sqrt{2}$ to obtain two quadratic equations, students often obtained $p, q = -4 \pm 3\sqrt{2}$ but confused which variable took which sign, or even used both possibilities in each case, leading either to incorrect signs on all four roots or to eight roots of which four were spurious. We also saw a significant number incorrectly taking $s = 2$. Most people who got to the end were very good at getting rid of the nested square roots using the method of part (i).

Question 5

It is clear that either vectors questions are not very popular, or that the topic itself is found difficult. Almost a quarter of the overall entry of 2000+ made a start at this question, most of which candidates (70% of them, in fact) attempted the opening explanation (usually very poorly) and then gave up. As a result, Q5 drew the lowest mean score of any of the paper's 11 questions, dipping down to a miserable 3/20 on average. Indeed, of the very few candidates who scored high marks on this question, most attacked it with more advanced (further maths) methods than are currently within the scope of the 9465 syllabus. It is, therefore, very difficult to say what it is that candidates did well, or did poorly, at.

To begin with, very few had a sufficient grasp of the properties of quadrilaterals and were unable to identify the parallelogram and rhombus of part (i). The only upside to this poor start for many was that it did help re-direct their attentions towards other questions. The beginning of (ii) then deterred further progress for most of the remaining candidates who had little idea as to what to do: the most "on-spec" approach being to check that the two diagonal lines intersect.

In (ii) (b), most good efforts used the scalar product, though the equivalent approach using the *Cosine rule* was the one intended (given that the SP is not actually on this part of the syllabus).

Question 6

As with several of the questions on the paper, the “start” supplied in the question gave many candidates the prospect of a grip on the content of the question, although – as was frequently the case – a significant proportion of starts petered out relatively quickly. Almost half of all candidates attempted this question, but then around a quarter of them fell by the wayside before making any substantial progress.

Many candidates preferred to find extrema by differentiation. Such efforts were rewarded for the quartic, but not for the quadratic since that part of the question required candidates to complete the square. Common mistakes followed from incorrectly dealing with the coefficient 9, and often candidates obtained different extrema for the two polynomials and were unable to make further progress. In many cases, candidates were unable to use the completed square form to find a minimum, while those who used differentiation often neglected to check the nature of the stationary points.

When sketching the graph, candidates often worked backwards from the inequalities given in the question to find turning points and did not receive many marks. The argument showing $\frac{\sin^2\theta\cos^2 x}{1+\cos^2\theta\sin^2 x} \leq 1$ was overcomplicated by many, though many approaches were successful. While some candidates could justify $\sin^2\theta = 1$ and $\cos^2 x = 1$ with clarity, many struggled for a cogent argument.

Question 7

Once again, the prospect of gaining an easy few marks at the question's opening drew in a lot of interest; but many of these attempts (a third of them, in fact) were of negligible success.

For serious takers, there was a good spread of marks, with some sensible comments and explanations being offered in several places. In part (i), steps 1, 3 and 4 were generally done well, although a common mistake in step 4 was to obtain an expression for b^2 that contained a "factor" of 3 but in which the other factor was not obviously an integer. Having shown step 4, however, a great many candidates thought that the contradiction obtained if a is a multiple of 3 was sufficient to conclude that $\sqrt{2}+\sqrt{3}$ was irrational, and didn't consider the case where a is not a multiple of 3, thus losing most of the marks for step 5. Some of those who avoided this pitfall forgot to appeal to symmetry to rule out the case of b being a multiple of 3. It was sometimes difficult to interpret candidates' logic from their prose.

Part (ii) was less well done in general, with many candidates giving very little detail. When considering squares of non-multiples of 5, a common error was to deal with the case $5k \pm 1$ but not $5k \pm 2$. A significant number guessed the correct relationship $a^4 + b^4 = 26a^2b^2$, but no credit was given for simply writing this without any reasoning. Very few candidates got the final mark for explaining why divisibility by 3 was not sufficient for this case, although some mentioned the key fact that 26 is 2 more than a multiple of 3, without saying why that is relevant.

Question 8

This was a fascinating question to mark, particularly as many candidates couldn't seem to figure out what to do with integrals that they *weren't* required to integrate. Another confusing feature of the question for many was the rather different role played by the variable x … a lot of candidates had clearly not encountered the notion that x doesn't have to be the (dummy) variable of the integrand; even amongst those who did make good progress with the question, checking what happened to the upper limit was a major stumbling-block. Almost 850 candidates attempted this question, 350 of whom failed to score more than 2/20 on it.

Those candidates who realised that this was a test of substitution integration generally coped very well with parts (i) and (ii), where the change-of-variable was more obvious; indeed, (ii) fell very readily to either one of two obvious tactics. Part (iii) required either a very keen appreciation of what was going on in the background or a careful build-up to the problem that started generally and then compared the result with the desired outcome. Quite a few candidates seemed to resort to guesswork and the accompanying working could often be vague, at best.

Part (iv), requiring a quadratic substitution, proved a step too far for many candidates, who – in spirit if not on paper – gave up at this point. Many appreciated that a quadratic substitution was 'on the cards' but couldn't make the conceptual leap of turning the $u^2\,\mathrm{d}u$ involved into $\sqrt{u^2}.\,u\,\mathrm{d}u$ so that the integrand could now be made to look more like the one in part (ii)'s integral.

Overall, full attempts were usually accompanied by high marks.

Question 9

There were fewer than 500 starts at Q9, with almost two-thirds of these attempts earning no more than 2/20 overall. Very few candidates were able to make a substantial attempt at this question; most were stymied by not having any physical intuition regarding the direction and position of the reaction forces at the moment of toppling. Once this was done then the question became mainly angle chasing and taking moments about appropriate points but most candidates lacked the confidence to make much progress.

Even the opening result in (i) – which required nothing more than taking moments (once) about a suitably chosen point – offered four marks for obtaining a given answer; this, however, proved too much for the majority of candidates. The real obstacle lay in the widespread reluctance among candidates to draw a good diagram, of a suitable size for labelling all necessary points, angles and forces (including their directions), and clearly labelled; a good diagram is *the* most important thing that a candidate can do … everything else follows so much more easily from a good diagram. In principle, all that is required to complete this question is to resolve twice, take moments (twice) and then apply the '*Law of Friction*' in its $F \le \mu R$ form and then sort out the resulting mix of algebra and trigonometry.

Question 10

This mechanics question was poorly done, despite being on the topic most frequently occurring on the early STEPs. Many candidates knew a general result for the trajectory of a projectile, but some could not adapt it when the angle given was with the vertical. Candidates did not seem to be sufficiently familiar with the sums and products of roots of a quadratic, which is new to the syllabus.

This question introduced two inequalities, and this is always a problem for candidates. The first came from the quadratic discriminant and the second came from the given point being not greater than the maximum height. However, candidates often resorted to algebraic meandering rather than clear thinking about how an inequality might arise.

Those candidates who really engaged with the question generally did well, although very high marks were seldom acquired.

Question 11

A relatively small number of candidates attempted this question, which was generally not well done. In part (i), candidates often failed to explain clearly how to get the given answer, with some simply calculating the probability for the first few values of n explicitly and spotting a pattern. Many candidates identified the given value as corresponding to $p = 0.5$, but lost marks for failing to explain why the probability was minimised at this point. Successful approaches to this part included differentiating, completing the square, or the AM-GM inequality.

Most candidates struggled with the setup for part (ii). Most got the correct probability of finishing in the first round, but generally didn't properly account for the different cases in the second round depending on the first round tosses. One occasional misconception was to think that the process is similar to part (i), i.e. that all three coins are tossed again if the first round is inconclusive. This incorrect approach fortuitously produces the same probability as a correct approach, so candidates who successfully analysed this situation could still get most of the marks. Again, relatively few candidates explained clearly why $p = 0.5$ gives the minimal probability.

STEP 1 Mathematics

(9465), 2019

Hints and solutions

Introductory Remarks

This document should be read in conjunction with the corresponding mark scheme in order to gain full benefit from it. Since the complete solutions appear elsewhere, much of this Hints and Solutions document will concentrate more on the "whys and wherefores" of the solution approach to each question and less on the technical details.
The solutions that follow, presented either in outline or in full, are by no means the only ones, not even necessarily the 'best' ones. They are simply intended to be the ones that, on the evidence of the marking process, appear to be the ones which arose most frequently from the ideas produced by the candidates and that worked for those who could force them through to a conclusion. If you "see" things in a different way, I hope you can still both follow and appreciate what is given here.

<u>Question 1</u>

Despite the fact that this differs from the usual sort of routine A-level question of this kind – in that there are no "numbers" involved in the presentation of it – the underlying ideas are, essentially, unchanged. The opening part contains a line with a given (trigonometric) gradient, which leads to a (right-angled) triangle with sides of (non-numerical) lengths. The vertices and area of this triangle are thus 'write downs' and the obvious approach in part (i) of using calculus to determine the minimum value of A should be clear to all.

(Now that there is no Formula Book available, it is important that candidates have learnt the various relationships – or can find them out speedily by hand if not – between the trig. functions and their derivatives, including the possibilities for deploying various trig. identities along the way, when necessary, in order to tidy any answers up.)

In Part (ii), the length of the hypotenuse, XY, of this same triangle needs to be determined in some form and, in principle at least, this is just a GCSE-level use of *Pythagoras' theorem*. Finding the perimeter of the triangle and, again, using calculus to maximise it consists of a set of well-established routines; it is only the accompanying trig. work that requires a bit of skill, some background learning of the results, and a certain amount of care.

Question 2

Since the first curve, C, is given in parametric form, candidates should be guided along the path of using the *Chain rule* result for parametric differentiation, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$, rather than reverting to finding its cartesian equation, hence avoiding issues with plus/minus signs that will clutter up the problem. $It\ is\ also\ important\ to\ think\ carefully\ about\ the\ difference\ between\ \ t$ and p: one is a general descriptive parameter for the curve while the other represents a particular value of it, even though this, too, is non-specific.

When finding the intersection, it is useful to simplify the expression using the difference of two cubes: $p^3 - q^3 \equiv (p - q)(p^2 + pq + q^2)$. This is the very useful cousin of the difference-of-two-squares factorisation. (Students might also like to look up the sum-of-two-cubes, the difference-of-two-fourth-powers, and similar results, for use in other problems … these are additional results often **not** flagged up automatically in STEPs but occurring sufficiently frequently to warrant learning in advance.) Once the intersections are in the required form, the constraint that the two tangents are perpendicular can now be used. Note the usefulness of the result that $p^2 + q^2 = (p + q)^2 - 2pq$ … another instance of a simple algebraic result that won't necessarily be sign-posted but which candidates *should* have in their 'toolkit' if they are to work through STEP questions fluently, rather than having to keep stopping to do odd bits of working *on one side* before being able to continue fluently with the solution to a problem.

There are various ways of finding the intersection of the two curves. One could turn everything into cartesian form, or just compare the parameterised x and y coordinates. It is tempting to try to use $u = p + q$ again, but that adds an additional constraint which will not necessarily hold at the point of intersection. We should end up with a cubic (or a disguised cubic) which needs to be factorised by first "spotting" a solution and then using the *factor theorem* and polynomial division. (Don't forget to check which of the solutions are valid … this is another routine STEP skill)

It should then be seen that one of the solutions is a double root – and this says something very significant about the intersection point: what? This should help with the final sketch. It is important to try to make sure that the sketch is consistent with all the information that has been given or found – for example, what is known about the gradient of C close to $x = 0$? What about the symmetry of C in the x axis?

Question 3

This question has a nice structure: firstly, an explicit "use this method" problem; followed by a "think about what's required for yourself, but it should be a similar kind of thing to the first bit" problem; with a "you're on your own now" finale.

As with Q1, there's a lot of different ways to go about these integrations, all governed by how readily one recognises what one has on the page at any stage of the process and on how easily it can be turned into something that can be integrated. A mixture of methods can be applied here, from direct integration, to use of the *Chain rule* in reverse (often referred to as "recognition" integration), through to the use of any one of a number of possible substitutions, or possibly even integration-by-parts.

In the first case, working on the integrand in the suggested way leads from

$$\frac{1}{1+\sin x} \text{ to } \frac{1-\sin x}{1-\sin^2 x} = \frac{1-\sin x}{\cos^2 x} = \sec^2 x - \sec x \tan x$$

and both of these terms in the final expression are directly integrable (being "standard" derivatives of $\tan x$ and $\sec x$ respectively).

The obvious approach (using a similar idea to that used for the first integral) to the second integral is to multiply top and bottom by $(1 - \sec x)$, and this leads from

$$\frac{1}{1+\sec x} \text{ to } \frac{1-\sec x}{1-\sec^2 x} = \frac{1-\sec x}{\tan^2 x} = \cot^2 x - \operatorname{cosec} x \cot x$$

and this is definitely *similar* to the situation that arose in the first case, but now requires the extra step of replacing $\cot^2 x$ with $(\operatorname{cosec}^2 x - 1)$ in order to get all terms in a directly integrable form. Alternatively, one could start the ball rolling here by turning

$$\frac{1}{1+\sec x} \text{ into } \frac{\cos x}{\cos x+1} \text{ or even } \frac{1+\cos x-1}{\cos x+1} = 1 - \frac{1}{\cos x+1}$$

(some students favour turning all trig. functions into sine and cosine only, and this approach can work as well) and then multiplying top and bottom by $(1 - \cos x)$. In fact, this second approach yields working which ends up looking (essentially) *exactly* like the first case, only with the extra manipulation work having been done upfront.

For the third integral, the most helpful method is not immediately obvious and one may need to be prepared to explore various ideas before full progress can be made. In fact, the initial idea used in this question would suggest that the best approach is to multiply top and bottom by $(1 - \sin x)^2$, and this turns out to be the right thing to do, although one must then be prepared to split the result into (possibly) several distinct parts and work on them separately. (As an aside, it is surprising how helpful it is to do this … almost all candidates who try to work on "the whole thing" in such cases end up making a mess of things, even if it is only by muddling a negative sign or missing a factor outside a bracket somewhere along the line.) I would direct the reader to the Mark Scheme, which shows how this third integral ends up being split (using the method suggested) into a direct integration, a substitution integration, and an integration-by-parts.

Question 4

Part (i) is clearly a simple start for later reference – as it doesn't look as if it turns up very soon from what can immediately be seen of part (ii) – and you are helpfully directed to consider integers from the outset. Since it is clearly a case of dealing with small numbers, it really should be possible to spot immediately that the (positive) square-root of $3+2\sqrt{2}$ can only be $1+\sqrt{2}$. If this is not immediately clear, then there is always the direct approach:

> expand $\left(m+n\sqrt{2}\right)^2$ to get $m^2+2n^2+2mn\sqrt{2}$; then compare the two parts with the known answer (hopefully avoiding a full algebraic method involving solving a quadratic-plus-linear pair of simultaneous equations, to find that $m=n=1$ works.

(But candidates are strongly advised to look out for every opportunity **not** to spend lots of time doing unnecessary algebra.)

Part (ii) begins with a little bit of an explanation, and this sort of thing must always be addressed as completely as possible – very many students tend to gloss over important details when it is important to be fully convincing.

Thereafter, multiplying out $(x^2+sx+p)(x^2-sx+q)$ and equating the coefficients with those of $\mathrm{f}(x)$ gives three equations for p, q and s. These can most helpfully be written in the following forms (suggested by the **given** form of the "Show that …" equation in the question):

$$(p+q)^2=(s^2-10)^2,$$
$$(p-q)^2=\frac{144}{s^2},$$
$$4pq=-8,$$

and it is clear that p and q are to be eliminated from this "system" of equations. Noting that the two squared terms on the left-hand-sides have a useful difference which leads to $(p+q)^2=(p-q)^2+4pq$ and rearranging then gives the required equation for s.

Substituting $t=s^2$ produces a cubic equation and the *factor theorem* provides us with a method for finding the three possible values of s^2 and we can soon find one solution, $t=2$. Now take out the corresponding factor, and solve a quadratic equation for the other two possible values.

If $s=\sqrt{2}$ (using the question's directing hint towards the use of the smallest value of s^2) the usefulness of the opening result should be revealed, and we now have the simultaneous equations for p and q: $p+q=-8$, $q-p=6\sqrt{2}$ … and now the two quadratic equations $x^2+sx+p=0$ and $x^2-sx+q=0$ can be solved separately. Each of the solutions will involve a term of the form $\sqrt{18\pm12\sqrt{2}}$, and the fact that $\sqrt{3+2\sqrt{2}}=1+\sqrt{2}$ from (i), together with a similar expression for $\sqrt{3-2\sqrt{2}}$, can be used to simplify the roots.

<u>Question 5</u>

Once again, there are two opening specific questions which require little more than a knowledge of the properties of quadrilaterals and the meaning that accompanies the statement that two vectors are equal. As with Q4, this is for later reference.

The start of part (i) can be approached in so many different ways, depending upon the depth of one's knowledge of further vector methods (where the equations of planes, independence of vectors, the vector product, the scalar triple product, etc. *could* all be considered). However, for knowledge of vectors at single maths level only, the obvious tactic is to show that the two diagonals intersect and this can be done using the standard sorts of method for finding the vector equations of the lines PR and QS and showing that they do indeed intersect provided the given condition holds.

Part (a) may seem unfamiliar territory, but the question gives all the necessaries, including the definition of the centroid of a quadrilateral. Note that an "if an only if" proof consists of two parts: the *if* ($\Rightarrow$) direction of the argument **and** the *only if* ($\Leftarrow$) part. Alternatively, in simple cases, it may be clear that any steps taken in the reasoning are entirely reversible ($\Leftrightarrow$) but this still needs to be stated clearly and not just left to be invisibly implied.

In (b), once one has now realised that $PQRS$ is a rhombus, one can proceed by working with the magnitudes of the relevant vectors (i.e. the lengths of the sides of the quadrilateral), and by using the immediately preceding result, in order to replace q, r and s with p. Then, to wrap things up (with the scalar product not now required for Paper 1) the *Cosine rule* can be used in triangle PQR to obtain the displayed result … thereafter, moving from rhombus to square by setting $\cos PQR = 0$. At the very end, there is a simple bit of inequality work to demonstrate the final given answer.

Question 6

From its appearance, one's initial impression of Q6 is that there's rather a lot to it … and, in fact, this does turn out to be the case. Nonetheless, the two parts of (i) are quite straightforward in terms of clearly stated demands and can be approached by everyone. The process of completing the square is relatively routine, even though one must be prepared to think of $\cos\theta$ as a numerical part of the coefficient of x (etc.) It is possible to factor out the 9 from the leading terms before commencing the process, but not really necessary, since the first two terms (to all intents and purposes the constant term, 4, at the end can be dealt with separately) will not require the use of fractions:

$$9x^2 - 12x\cos\theta \equiv (3x - 2\cos\theta)^2 - \text{an adjusting constant.}$$

The final, overall adjusted, constant turns out to be $4 - 4\cos^2\theta$ and it is helpful to replace this immediately by $4\sin^2\theta$. The purpose of the completed-square form for the original expression is that the minimum value, and the value of x which gives it, can then be written down without further ado.

The second of these given expressions is a quadratic in (x^2) and one could complete the square (effectively hinted at from the first request) or – since there is now no clear direction as to which method should be used – find the required maximum (also $4\sin^2\theta$) using calculus. The summary result of part (i) now follows by separating off the two bits previously re-formatted and realising that the minimum of one side can only hit the maximum of the other – in this case – at the one instant, $4\sin^2\theta$.

Setting the x's equal and then turning the resulting equation into a quadratic in ($\sin x$) leads to the single solution $\sin x = \frac{1}{2}$. Now, this is one of the "standard" exact trig. results which all candidates should recognise, though it does lead to two values of x in the given interval (right back at the very start of the question) of $0 < x < \pi$. It is then important both to consider the positive *and* negative square-roots of the values for x AND to remember that, having squared along the way, there are likely to be "extraneous solutions" that shouldn't be there and hence need to be checked for validity at the end.

In part (ii), it soon becomes clear that the given curve has two branches, a $\cup$-shaped bit and a $\cap$-shaped bit. A standard calculus approach (helped by a consideration of what happens as $x \to 0\pm$ and as $x \to \pm\infty$) shows that there is a minimum at $(2\theta, 4\theta)$ and a maximum at (0, 0), as required. Next, in the given rational expression, it can be quickly noted that the numerator is always less than or equal to 1 while the denominator is greater than or equal to 1, all of which gives us the desired result. Finally, re-arranging the second of the initial two established inequalities, and setting it alongside the next one gives

$$\frac{x^2}{4\theta(x-\theta)} \geq 1 \geq \frac{\sin^2\theta\cos^2 x}{1+\cos^2\theta\sin^2 x}$$

and, with the same notion as appeared in part (i), the two can only be equal when top and bottom both take their maximum/minimum values respectively.

Question 7

This is very much a "reasoning" question, where the explanations form a major part of the deal and there is no point skimming through these and expecting to come out with high marks. Although it is not essential to have some grasp of the ideas behind modular arithmetic, the notations used are immensely brief compared to the written word … so, for instance, the statement

$$x \equiv 2 \pmod 3$$

is actually saying that the number (integer) x leaves a remainder of 2 upon division by 3; as such, this is a statement being made about an infinite set of numbers which have such a property, without worrying about the actual whole number part when the division is undertaken (a bit like the *Remainder theorem*). It is also worth noting that we can switch between positive and negative "remainders" without any need for additional explanation; and $x \equiv 2 \pmod 3$ is, in fact, exactly the same as saying that $x \equiv -1 \pmod 3$ since saying a number is 2 more than one multiple of 3 is equivalent to saying that it is 1 less than another (the next, in fact).

So, with this basic notation in mind, let's try to avoid using it as far as possible (though it appears within the published MS to some extent). To begin with part (i):

Step 1. If a is not a multiple of 3, dividing by 3 leaves a remainder of 1 or 2, so then $a = 3j + 1$ or $a = 3j - 1$. Squaring then gives $a^2 = 9k^2 \pm 6k + 1 = 3(3k^2 \pm 2k) + 1$, which is clearly **shown** to be 1 more than a multiple of 3.

Step 3. We have $\left(\frac{a}{b}\right)^2 = \left(\sqrt{2} + \sqrt{3}\right)^2 = 5 + 2\sqrt{6}$, and $\left(\frac{a}{b}\right)^4 = \left(\sqrt{2} + \sqrt{3}\right)^4 = 49 + 20\sqrt{6}$. So we then have $\left(\frac{a}{b}\right)^4 = 10\left(\frac{a}{b}\right)^2 - 1$, a relationship clearly suggested by the result to which we are working, and this can then be rearranged to the desired form.

Step 4. Writing $a = 3k$ and rearranging, $b^4 = 90k^2b^2 - 81k^4$, which is clearly a multiple of 3. So b must also be a multiple of 3.

Step 5. It follows that if a is a multiple of 3 then a and b have 3 as a common factor, contradicting the assumption of step 2. This is not yet enough to conclude that $\sqrt{2} + \sqrt{3}$ is irrational, since it is also necessary to deal with the case when a is not a multiple of 3. If b is a multiple of 3, then so is a, by symmetric reasoning, so the final case is when neither is a multiple of 3. In this case we use Step 1: a^4, b^4 and a^2b^2 are all squares of numbers which are not multiples of 3, so each of these is 1 more than a multiple of 3. It follows that $a^4 + b^4$ is 2 more than a multiple of 3, but $10a^2b^2$ is 1 more than a multiple of 3, a contradiction.

In part (ii), it should be clear that a similar type of working will be employed **but** there is going to be at least one significant difference (see the comment required at the very end) so one must be careful to notice things work slightly differently in this slightly new situation.

As expected, then, this works similarly to part (i), but is more complicated. Numbers which are not multiples of 5 can have the form $5k \pm 1$ or $5k \pm 2$. Squaring each of these separately, their squares all have the form $5m \pm 1$, and the fourth powers all have the form $5n + 1$.

Expanding powers of $\sqrt{6} + \sqrt{7}$ and proceeding as for Step 3 of part (i) gives the relationship $a^4 + b^4 = 26a^2b^2$. If either a or b is a multiple of 5, a similar argument to part (i) shows that a and b have a common factor of 5, contradicting the assumption on a and b. If not, we can use what we know about squares and fourth powers to argue that the left-hand side is 2 more than a multiple of 5 but the right-hand side is either 1 more than, or 1 less than, a multiple of 5, giving a contradiction.

As a final thought, it is easy to demonstrate that a contradiction purely by considering multiples of 3 is not possible. This is because 26 is 2 more than a multiple of 3, so if a^2 and b^2 are each 1 more than a multiple of 3, both sides of the equation will be 2 more than a multiple of 3.

Question 8

This question deals with the process of substitution integration itself and how it can be used to show how things are related *functionally*.

In order to be entirely comfortable with this, one must first realise the roles being played by the various letters. At this level, it should be clear that

$$\int_1^2 \mathrm{f}(x)\,\mathrm{d}x = \int_1^2 \mathrm{f}(y)\,\mathrm{d}y = \int_1^2 \mathrm{f}(t)\,\mathrm{d}t \text{ etc.}$$

since the letter being used within the integrand is irrelevant to whatever is going on … it is called a *dummy variable* and is only there as an indicator. (Of course, these letters may contain within them a wider, geometric or graphical, significance to the whole integral, such as indicating the area between the curve and the x- or y-axes in the first two cases above; but the result itself – a number – is unchanged by the choice of letter. So, when one sees the given definition of f(x) in integral form, the upper limit of x is what now guarantees that the answer is indeed a function of x rather than of t and it has a rather different role.

For (i), the $\mathrm{f}(\frac{1}{2}x)$ in the given answer gives the big hint: setting $t = \frac{1}{2}u$ in the original integral should (and does) do the trick … but it is still very important to **show** that everything works out properly, such as demonstrating how the limits change from those for t to those for u.

In (ii), one can either take up where (i) finished … setting $v = u - 2$ (say) … or go back to the original integral and set $v = 2t - 2$, again being careful to show how everything works out.

In part (iii), the first real challenge is presented. What must be done *can* be seen intuitively, but there is also the perfectly sensible tactic – also requiring insight – that a linear substitution must be required, but it is not immediately clear what it is. So, try setting $u = at + b$ and forcing it through, then comparing the outcomes with what is needed. Doing this reveals that $a = 3$ and $b = 2$ do the trick admirably.

Having decided that a linear substitution worked in (iii), the jump now for part (iv) would seem to require a quadratic substitution. Even something simple like $y = u^2$ would give $\mathrm{d}y = 2u\,\mathrm{d}u$ in the substitution working and, at first glance, though it looks as if the numerator of the final integrand is going to cause trouble, we can write $\int \frac{u^2}{\sqrt{u^2+4}}\,\mathrm{d}u$ as $\int \sqrt{\frac{u^2}{u^2+4}}\,u\,\mathrm{d}u$ and it is now seen that the integrand is now starting to look remarkably similar to the form of (ii)'s integral, namely $\int \sqrt{\frac{X}{X+4}}\,\mathrm{d}X$, which has already been addressed. It is only the remaining details that must be dealt with, along with the fact that the final answer is not just a single f(--) thing.

Question 9

As with many mechanics questions, a sensibly large diagram is important; if not actually essential, then at least a considerable advantage. In this case, it is best to consider a 2-d vertical "cross-section" through ladder and box. There are really three bits of physical insight required when setting up this diagram. The first is that the reaction force between the box and the ladder acts at right angles to the ladder. The second is that, if the box topples, then it will rotate around the edge diagonally opposite the contact point between the ladder and the box. The third is that at the point of toppling the normal force from the ground on the box will act through this tipping point.

Once the diagram has been set up suitably, the question boils down to choosing which points to take moments about or which direction(s) to resolve in.

For part (i) we want to link the painter's mass and the reaction force, so taking moments about the point of contact of the ladder and the ground seems most sensible. Note also, that there is no interest in what happens to the ladder … taking moments about any other point would require a consideration of what is going on at the foot of the ladder.

This has not really brought in anything to do with the box toppling, so if we are looking for another crucial equation we should take moments about one of the base corners of the box. This is what is required in part (ii). The geometry of this can be quite tricky to deal with – the reaction force is not pointing in a convenient direction to deal with directly. However, some angle chasing will allow you to resolve it into vertical and horizontal components.

The final part brings in friction, so we will make use of the fact that the box topples before it slides so the friction force must not have gone past its maximum possible value. This means that $F \leq \mu N$. Resolving vertically and horizontally gives enough information about F and N to make progress with this expression, but then it is necessary to use some of the previous parts (as so often in STEP!) and some trigonometric identities to form the required expression.

Question 10

Although not essential in this question, it is still a good idea to begin with a clear diagram; especially since there is a slight twist to the traditional set-up here as the angle given is with the vertical (not the horizontal), so standard results need to be modified carefully to fit.

Given this slight variation, it is best in part (i) to derive the usual *trajectory equation* rather than "quote it", though there is no harm in the latter approach: remember that t plays no part in this equation so we find a simple equation involving t, rearrange it and eliminate t in the other equations. Then, substituting in $y = h$ when $x = h\tan\beta$, and making use of some trig. identities, leads to the required answer.

In part (a), it should be clear that one can use the results for the sum of the roots of a quadratic. This might also suggest the idea that the product of the roots may be useful later on. We now have lots of information involving cots so you might think that, in order to show that $\alpha_1 + \alpha_2 = \beta$, one needs only to track back to $\cot(\alpha_1 + \alpha_2) = \cot\beta$, and this indeed turns out to be the case. Although the compound angle formula for cot is not commonly used, it is easily derived from the compound angle formula for tan. There is one technical issue here, however, which is this: just because $\tan A = \tan B$ it does not follow that $A = B$. For example, $\tan 0 = \tan 180$. You will have to come up with an argument about why we can equate the arguments of the cot function in this case.

Up to this point we've not really made use of the fact that there are two solutions. Looking back, it is seen that the equation (*) is a quadratic in c so there is a common method for determining how many solutions it has.

In part (ii), an expression for the greatest height is needed, but we must be careful that u and α are fixed in the question, so we are looking for the greatest height over the time of flight. This will occur when the vertical velocity is zero. We can then get the required inequality by comparing this value to the known achieved height h.

Question 11

In the situation described in part (i), in each round, a decision is made if the coins are the same way up (with probability $p^2 + q^2$) and the process continues if they are different (with probability $pq + qp = 2pq$). So the probability that they continue $(n-1)$ times and then make a decision on the nth round is $(2pq)^{n-1}(p^2 + q^2)$.

The probability that they don't make a decision on the first n rounds is $(2pq)^n$, and so the probability they do is $1 - (2pq)^n$. Now $2pq = 2p - 2p^2$, and completing the square will show that this is at least $\frac{1}{2}$, giving the required bound.

In part (ii), a decision is made in the first round if all coins are the same; i.e. with probability $p^3 + q^3$. In order to make a decision on the second round, there are two possibilities. One possibility is that the first round has two heads and a tail, the tail is turned over to become a head, and the other two coins are tossed, both resulting in heads. This has probability $3p^4q$. The other possibility is the same with heads and tails exchanged, so has probability $3pq^4$.

So we wish to find the minimum value of $p^3 + q^3 + 3p^4q + 3pq^4$. One way to do this is to write it as a function of p only, using the fact that $q = 1 - p$, and to differentiate. If done correctly, this will give three turning points at $p = 0$, $p = \frac{1}{2}$ and $p = 1$. By considering the second derivative we find that $p = 0$ and $p = 1$ are local maxima, but $p = \frac{1}{2}$ is a local minimum; so it follows that the minimum value for $0 \le p \le 1$ is at $p = \frac{1}{2}$, and evaluating the function at this point gives $\frac{7}{16}$.

STEP MATHEMATICS 1

2019

Mark Scheme

1	Eqn. of line is $y - k = -\tan\theta(x - 1)$ or $y + x\tan\theta = k + \tan\theta$		
	Eqn. of line found with substn. of $y = 0$, $x = 0$ in turn	**M1**	
	$X = (1 + k\cot\theta, 0)$ and $Y = (0, k + \tan\theta)$	**A1 A1**	**3**
	(i) $A = \frac{1}{2}(OX)(OY) = \frac{1}{2}(1 + k\cot\theta)(k + \tan\theta)$		
	$= \frac{1}{2}\left(k^2\cot\theta + 2k + \tan\theta\right) = \frac{1}{2\tan\theta}(k + \tan\theta)^2$	**B1 ft** (any correct form)	
	$\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{1}{2}\left(-k^2\mathrm{cosec}^2\theta + \sec^2\theta\right)$	**M1 A1** for the differentiation	
	or $\frac{\mathrm{d}A}{\mathrm{d}\theta} = \frac{1}{2\tan\theta}2(k + \tan\theta)\sec^2\theta - \frac{1}{2\tan^2\theta}\sec^2\theta(k + \tan\theta)^2$		
	$= \frac{(k + \tan\theta)\sec^2\theta}{2\tan^2\theta}\{2\tan\theta - (k + \tan\theta)\} = \frac{\sec^2\theta(\tan\theta + k)(\tan\theta - k)}{2\tan^2\theta}$		
	$= 0$ when	**M1** derivate set = 0 and solved	
	Either $\tan\theta = -k$ ($\Rightarrow$ $A = 0$, but rejected since $\tan\theta > 0$ in given region)		
	(Not necessary to mention this explicitly)		
	or $\tan\theta = k \Rightarrow A = 2k$	**A1**	**5**
	(ii) $XY = 1 + k\cot\theta + k + \tan\theta + \sqrt{(1 + k\cot\theta)^2 + (k + \tan\theta)^2}$	**M1** for attempt at XY	
	$= 1 + k\cot\theta + k + \tan\theta + (k + \tan\theta)\sqrt{\cot^2\theta + 1}$		
	NB $XY\sin\theta = k + \tan\theta$ (e.g.) gives XY without distance formula		
	$L = OX + OY + XY = 1 + \frac{k}{\tan\theta} + k + \tan\theta + (k\mathrm{cosec}\theta + \sec\theta)$	**M1**	
	Use of relevant trig. identity to find XY without square-root and with all three sides involved		
	(No need to justify taking the +ve sq-rt. since given $0 < \theta < \frac{1}{2}\pi$)		
	$L = 1 + \tan\theta + \sec\theta + k(1 + \cot\theta + \mathrm{cosec}\theta)$	**A1** legitimately(**AG**)	**3**
	$\frac{\mathrm{d}L}{\mathrm{d}\theta} = k\left(-\mathrm{cosec}^2\theta - \mathrm{cosec}\theta\cot\theta\right) + \left(\sec^2\theta + \sec\theta\tan\theta\right)$	**M1**	
	$= 0$ when $k = \frac{\sec\theta(\sec\theta + \tan\theta)}{\mathrm{cosec}\theta(\mathrm{cosec}\theta + \cot\theta)}$	**A1**	
	$= \frac{\frac{1}{c}\left(\frac{1}{c} + \frac{s}{c}\right)}{\frac{1}{s}\left(\frac{1}{s} + \frac{c}{s}\right)} = \frac{\frac{1}{c^2}(1 + s)}{\frac{1}{s^2}(1 + c)} = \frac{s^2(1 + s)}{c^2(1 + c)}$	**M1** trig. method for getting k	
	$= \frac{(1 - c)(1 + c)(1 + s)}{(1 - s)(1 + s)(1 + c)}$	**M1** use of $c^2 + s^2 = 1$ etc.	
	$= \frac{1 - c}{1 - s}$	**A1** legitimately (**AG**)	
	Allow the final 3 marks for using the given answer to verify that $\frac{\mathrm{d}L}{\mathrm{d}\theta} = 0$ (provided that $\theta = \alpha$ used)		
			5

Then $L_{\min} = \left(\frac{1-c}{1-s}+\frac{s}{c}\right)\left(1+\frac{c}{s}+\frac{1}{s}\right)$ **M1** substituting back

Must use correct (given) expression for L

$$= \left(\frac{c-c^2+s-s^2}{c(1-s)}\right)\left(\frac{c+s+1}{s}\right)$$ **M1** common denominators

$$= \left(\frac{c+s-1}{c(1-s)}\right)\left(\frac{c+s+1}{s}\right)$$ **M1** for dealing with numerator

$(c+s)^2 - 1 = c^2 + s^2 + 2cs - 1$

$$= \frac{2cs}{cs(1-s)} = \frac{2}{1-\sin\alpha}$$ **A1** final answer (exactly this)

4

2	$\dfrac{\mathrm{d}y}{\mathrm{d}x}=\dfrac{\frac{\mathrm{d}}{\mathrm{d}t}\left(2t^3\right)}{\frac{\mathrm{d}}{\mathrm{d}t}\left(3t^2\right)}=\dfrac{6t^2}{6t}=t$ so grad. tgt. at $P(3p^2, 2p^3)$ is p	**M1 A1**	
	Eqn. tgt. at P is then $y-2p^3=p(x-3p^2)$ i.e. $y=px-p^3$	**M1 A1** legitimately (**AG**)	**4**
	$y=px-p^3$ meets $y=qx-q^3$ when $px-p^3=qx-q^3 \Rightarrow (p-q)x=(p^3-q^3)$	**M1** equating y's and rearranging for x	
	and since $p\neq q$, $x=p^2+pq+q^2$, $y=pq(p+q)$	**A1 A1** x, y must be simplified	**3**
	Tgts. perpr. iff $pq=-1 \Rightarrow u=p-\dfrac{1}{p}$, $u^2=p^2+\dfrac{1}{p^2}-2$	**M1 A1** seen or implied	
	and $P_1=\left(p^2+\dfrac{1}{p^2}-1,\ -\left[p-\dfrac{1}{p}\right]\right)=(u^2+1, -u)$	**A1** (**AG**) legitimately	**3**
	EITHER $x=y^2+1$ meets $\dfrac{x^3}{27}=\dfrac{y^2}{4}$ OR $\left(\dfrac{u^2+1}{3}\right)^3=t^6=\left(\dfrac{-u}{2}\right)^2$	**M1**	
	when $4(y^2+1)^3=27y^2$ $\quad 4(u^2+1)^3=27u^2$		
	$4(v^6+3v^4+3v^2+1)=27v^2$ $\quad (v=u$ or $y)$		
	Use of cubic expansion, incl. 1-3-3-1 coefficients	**M1**	
	$4v^6+12v^4-15v^2+4=0$		
	$(v^2+4)(4v^4-4v^2+1)=0$	**M1** attempt to find a factor	
	$(v^2+4)(2v^2-1)^2=0$	**A1** complete factorisation	
	$v^2\neq -4 \Rightarrow y^2=\frac{1}{2}$, (OR via $u=\mp\frac{1}{\sqrt{2}}$, $t=\pm\frac{1}{\sqrt{2}}$)	$y=\pm\frac{1}{\sqrt{2}}$, $x=\frac{3}{2}$	
	One for each (cartesian) coordinate	**A1 A1**	
	ALT.		
	$u^2+1=3t^2$ and $-u=2t^3 \Rightarrow 4t^6-3t^2+1=0$	**M1 A1** Eliminating u	
	$\Rightarrow (t^2+1)(2t^2-1)^2=0$	**M1 A1** attempt to factorise; correct	
	$\Rightarrow t=\pm\frac{1}{\sqrt{2}}$ $\quad y=\pm\frac{1}{\sqrt{2}}$, $x=\frac{3}{2}$	**A1 A1**	**6**
	<u>Graphs:</u>		
	C is a semi-cubical parabola with a cusp at O	**B1**	
	$\tilde{C}$ is a $\subset$-shaped parabola with vertex at (1, 0)*	**B1** (* apparently, here)	
	Curves meet tangentially	**B1**	
	Key points noted or sketched, esp. (1, 0) and contacts at $x=\frac{3}{2}$	**B1**	
	(Withhold final mark if unclear which curve is which)		**4**

3 **(i)** $I = \int_0^{\pi/4} \frac{1}{1+\sin x}\,dx = \int_0^{\pi/4} \frac{1-\sin x}{\cos^2 x}\,dx = \int_0^{\pi/4} \sec^2 x\,dx - \int_0^{\pi/4} \frac{\sin x}{\cos^2 x}\,dx$

M1 use of $1 - s^2 = c^2$ and splitting into two integrals

Now $\int_0^{\pi/4} \sec^2 x\,dx = [\tan x]_0^{\frac{1}{4}\pi} = 1$ **B1**

and $\int_0^{\pi/4} \frac{\sin x}{\cos^2 x}\,dx = \int_1^{1/\sqrt{2}} \frac{-1}{c^2}\,dc$ using substn. $c = \cos x$, $dc = -\sin x\,dx$ etc.

M1 (or by "recognition")

$= \left[\frac{1}{c}\right]_1^{\frac{1}{\sqrt{2}}} = \sqrt{2} - 1$ **A1**

OR via $\int \frac{\sin x}{\cos^2 x}\,dx = \int \sec x \tan x\,dx = \sec x$ (**M1 A1**)

so that $I = 2 - \sqrt{2}$ **A1 cao** **5**

(ii) $\int_{\pi/4}^{\pi/3} \frac{1}{1+\sec x}\,dx = \int_{\pi/4}^{\pi/3} \frac{\sec x - 1}{\tan^2 x}\,dx$ **M1** use of initial technique

$= \int_{\pi/4}^{\pi/3} \frac{\cos x - \cos^2 x}{\sin^2 x}\,dx = \int_{\pi/4}^{\pi/3} \frac{\cos x}{\sin^2 x}\,dx - \int_{\pi/4}^{\pi/3} \cot^2 x\,dx$ **M1** split appropriately

$= \int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{1}{s^2}\,ds - \int_{\pi/4}^{\pi/3} \left(\operatorname{cosec}^2 x - 1\right)dx$ **M1** use of relevant trig. identity

NB $\int \frac{\cos x}{\sin^2 x}\,dx = \int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x$

$= \left[\frac{-1}{s}\right]_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} + [\cot x + x]_{\frac{1}{4}\pi}^{\frac{1}{3}\pi}$ **A1 A1**

$= \left(-\frac{2}{\sqrt{3}} + \frac{2}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{3}} + \frac{\pi}{3} - 1 - \frac{\pi}{4}\right)$

$= \frac{\pi}{12} + \sqrt{2} - 1 - \frac{1}{\sqrt{3}}$ **A1 cao** in a suitable, exact form

6

ALT. I

$\int_{\pi/4}^{\pi/3} \frac{1}{1+\sec x}\,dx = \int_{\pi/4}^{\pi/3} \frac{\cos x}{1+\cos x}\,dx$ **M1**

$= \int_{\pi/4}^{\pi/3} \frac{1+\cos x - 1}{1+\cos x}\,dx = \int_{\pi/4}^{\pi/3} \left(1 - \frac{1}{1+\cos x}\right)dx$ **M1**

$$= \left(\frac{\pi}{3} - \frac{\pi}{4}\right) - J$$

Using the initial technique,

$$J = \int_{\pi/4}^{\pi/3} \frac{1-\cos x}{\sin^2 x}\,dx = \int_{\pi/4}^{\pi/3} \left(\operatorname{cosec}^2 x - \frac{\cos x}{\sin^2 x}\right)dx$$ **M1**

$$= \left[-\cot x\right]_{\frac{1}{4}\pi}^{\frac{1}{3}\pi} - \int_{\sqrt{2}/2}^{\sqrt{3}/2} \frac{1}{s^2}\,ds$$ **A1 M1** (full substn. attempt)

$$= \frac{-1}{\sqrt{3}} + 1 + \left[\frac{1}{s}\right]_{\frac{\sqrt{2}}{2}}^{\frac{\sqrt{3}}{2}} = 1 - \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{2}}$$

giving $\displaystyle\int_{\pi/4}^{\pi/3} \frac{1}{1+\sec x}\,dx = \frac{\pi}{12} + \sqrt{2} - 1 - \frac{1}{\sqrt{3}}$ **A1** **6**

ALT. II

$$\int_{\pi/4}^{\pi/3} \frac{1}{1+\sec x}\,dx = \int_{\pi/4}^{\pi/3} \frac{\cos x}{1+\cos x}\,dx$$ **M1**

$$= \int_{\pi/4}^{\pi/3} \frac{2\cos^2 \frac{1}{2}x - 1}{2\cos^2 \frac{1}{2}x}\,dx$$ **M1**

$$= \int_{\pi/4}^{\pi/3} \left(1 - \tfrac{1}{2}\sec^2 \tfrac{1}{2}x\right)dx$$ **M1**

$$= x - \tan\tfrac{1}{2}x$$ **A1**

$$= \left(\frac{\pi}{3} - \frac{\pi}{4}\right) - \frac{1}{\sqrt{3}} + \left(\sqrt{2} - 1\right) = \frac{\pi}{12} + \sqrt{2} - 1 - \frac{1}{\sqrt{3}}$$ **M1 A1**

The M is for a method to find $\tan\left(\frac{1}{8}\pi\right)$ **6**

(iii) $\displaystyle\int_0^{\pi/3} \frac{1}{(1+\sin x)^2}\,dx = \int_0^{\pi/3} \frac{(1-\sin x)^2}{\cos^4 x}\,dx$ **M1** multg. nr. & dr. by $(1 - \sin x)^2$

$$= \int_0^{\pi/3} \frac{1 - 2\sin x + \sin^2 x}{\cos^4 x}\,dx = \int_0^{\pi/3} \frac{2 - 2\sin x - \cos^2 x}{\cos^4 x}\,dx$$ **M1**

$$= \int_0^{\pi/3} 2\sec^4 x\,dx - \int_0^{\pi/3} \frac{2\sin x}{\cos^4 x}\,dx - \int_0^{\pi/3} \sec^2 x\,dx$$ **A1**

Must be separated into individually integrable forms *

Now $\displaystyle\int_0^{\pi/3} \frac{2\sin x}{\cos^4 x}\,dx = -2\int_1^{1/2} \frac{-1}{c^4}\,dc = \left[\frac{2}{3c^3}\right]_1^{\frac{1}{2}} = \frac{14}{3}$ **M1 A1**

and $\displaystyle\int_0^{\pi/3} \sec^2 x\,dx = \left[\tan x\right]_0^{\frac{1}{3}\pi} = \sqrt{3}$ -- Rewarded in final answer mark

$$K = \int_0^{\pi/3} \sec^4 x\,dx = \int_0^{\pi/3} \sec^2 x.\sec^2 x\,dx = \left[\sec^2 x.\tan x\right]_0^{\frac{1}{3}\pi} - \int_0^{\pi/3} 2\sec^2 x.\tan^2 x\,dx$$

M1 A1 use of *integration by parts*

$K = 4\sqrt{3} - 0 - 2\int_0^{\pi/3} \sec^2 x(\sec^2 x - 1)\,dx$ **M1** 'recognition' attempt with *loop*

$= 4\sqrt{3} - 2K + 2[\tan x]_0^{\frac{1}{3}\pi} = 4\sqrt{3} - 2K + 2\sqrt{3}$

$\Rightarrow K = 2\sqrt{3}$ -- Rewarded in final answer mark

giving $\int_0^{\pi/3} \frac{1}{(1+\sin x)^2}\,dx = 3\sqrt{3} - \frac{14}{3}$ **A1** **9**

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* NB $\int\left(\frac{1+s^2}{c^4}\right)dx = \int\left(\frac{c^2+2s^2}{c^4}\right)dx = \int \sec^2 x\,dx + \int 2\sec^2 x\tan^2 x\,dx = \tan x + \frac{2}{3}\tan^3 x$

---

4 **(i)** $\sqrt{3+2\sqrt{2}} = 1+\sqrt{2}$ by (e.g.) squaring and comparing terms in m, n

M1 A1 (i.e. $m = n = 1$)

NB A0 for $m = n = -1$ also **2**

(ii) Existence of four roots α, β, γ, δ means we must have

$f(x) = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)$

$= (x^2 - [\alpha+\beta]\,x + \alpha\beta)(x^2 - [\gamma+\delta]\,x + \gamma\delta)$ **E1** Justifying factorisation into quadratics

Since $\alpha+\beta+\gamma+\delta = 0$ from coefft. of x^3 in $f(x)$

it follows that $\alpha+\beta = -(\gamma+\delta)$ **B1**

Comparing the other coeffts. of $f(x)$ **M1** (or by multiplying out)

$pq = -2$

$s(p-q) = -12$ **A1** for at least 2 correct

$p + q - s^2 = -10$ **A1** for 3rd correct

Use of $p + q = s^2 - 10 \Rightarrow (p+q)^2 = (s^2-10)^2$

and $(p-q)^2 = (p+q)^2 - 4pq = \left(\dfrac{12}{s}\right)^2$ **M1**

to get an eqn. in s only **A1** (correct unsimplified)

$\Rightarrow s^2(s^2-10)^2 + 8s^2 - 144 = 0$ **A1** (**AG**) legitimately obtained **8**

$(s^2-10)^3 + 10(s^2-10)^2 + 8(s^2-10) - 64 = 0$ **M1** attempt at cubic in (s^2-10)

i.e. $u^3 + 10u^2 + 8u - 64 = 0 \Rightarrow (u-2)(u+4)(u+8) = 0$ **M1** finding one factor

A1 complete linear factorisation

$\Rightarrow s^2 - 10 = 2, -4, -8 \Rightarrow s^2 = 12, 6, 2$ **A1** (i.e. $\Rightarrow s = \pm 2\sqrt{3}, \pm\sqrt{6}, \pm\sqrt{2}$)

ALT.

$s^2(s^4 - 20s^2 + 100) + 8s^2 - 144 = 0$

$\Rightarrow s^6 - 20s^4 + 108s^2 - 144 = 0$ M1 attempt at cubic in s^2

$\Rightarrow (s^2-2)(s^2-6)(s^2-12) = 0$ M1 finding one factor

A1 complete linear factorisation

$\Rightarrow s^2 = 12, 6, 2$ A1 **4**

$s = \sqrt{2}$, $p = -4-3\sqrt{2}$, $q = -4+3\sqrt{2}$

(Note that taking the –ve sq.rt. simply swaps the brackets)

Or $s = \sqrt{6}$, $p = -2-\sqrt{6}$, $q = -2+\sqrt{6}$

Or $s = 2\sqrt{3}$, $p = 1-\sqrt{3}$, $q = 1+\sqrt{3}$

Candidates told to use the smallest value of s^2, so the working *should* proceed as follows:-

$f(x) = \left(x^2 + x\sqrt{2} - 4 - 3\sqrt{2}\right)\left(x^2 - x\sqrt{2} - 4 + 3\sqrt{2}\right) = 0$ **M1**

(using $s = \sqrt{2}$, $t = -\sqrt{2}$, $p = -4-3\sqrt{2}$, $q = -4+3\sqrt{2}$)

Using quadratic formula on each factor:

$x = \dfrac{-\sqrt{2} \pm \sqrt{18+12\sqrt{2}}}{2}, \dfrac{\sqrt{2} \pm \sqrt{18-12\sqrt{2}}}{2}$ **M1 A1** (**A** for 2 correct discriminants)

$$= \frac{-\sqrt{2} \pm \sqrt{6}\sqrt{3+2\sqrt{2}}}{2}, \frac{\sqrt{2} \pm \sqrt{6}\sqrt{3-2\sqrt{2}}}{2}$$

$$= \frac{-\sqrt{2} \pm \sqrt{6}\left(1+\sqrt{2}\right)}{2}, \frac{\sqrt{2} \pm \sqrt{6}\left(\sqrt{2}-1\right)}{2}$$

M1 using **(i)**'s result

$$x = \frac{-\sqrt{2}+\sqrt{6}+2\sqrt{3}}{2}, \frac{-\sqrt{2}-\sqrt{6}-2\sqrt{3}}{2}, \frac{\sqrt{2}+\sqrt{6}-2\sqrt{3}}{2}, \frac{\sqrt{2}-\sqrt{6}+2\sqrt{3}}{2}$$

A1 any two correct

A1 all four (& no extras) **6**

However,

$f(x) = \left(x^2 + x\sqrt{6} - 2 - \sqrt{6}\right)\left(x^2 - x\sqrt{6} - 2 + \sqrt{6}\right) = 0$

(using $s = \sqrt{6}$, $t = -\sqrt{6}$, $p = -2-\sqrt{6}$, $q = -2+\sqrt{6}$)

with discriminants $14 + 4\sqrt{6} = 2\left(7 + 2\sqrt{6}\right) = \left[\sqrt{2}\left(1+\sqrt{6}\right)\right]^2$ and $14 - 4\sqrt{6}$ etc.

and $f(x) = \left(x^2 + x\sqrt{12} + 1 - \sqrt{3}\right)\left(x^2 - x\sqrt{12} + 1 + \sqrt{3}\right) = 0$

(using $s = 2\sqrt{3}$, $t = -2\sqrt{3}$, $p = 1-\sqrt{3}$, $q = 1+\sqrt{3}$)

with discriminants $8 + 4\sqrt{3} = 2\left(4 + 2\sqrt{3}\right) = \left[\sqrt{2}\left(1+\sqrt{3}\right)\right]^2$ and $8 - 4\sqrt{3}$ etc.

5 **(i)** If $\overrightarrow{PQ} = \overrightarrow{SR}$ then $PQRS$ is a parallelogram **B1**

If $\overrightarrow{PQ} = \overrightarrow{SR}$ and $\left|\overrightarrow{PQ}\right| = \left|\overrightarrow{PS}\right|$ then $PQRS$ is a rhombus **B1** **2**

$$\overrightarrow{PQ} = \begin{pmatrix} 1-p \\ q \\ 0 \end{pmatrix}, \quad \overrightarrow{PR} = \begin{pmatrix} r-p \\ 1 \\ 1 \end{pmatrix}, \quad \overrightarrow{PS} = \begin{pmatrix} -p \\ s \\ 1 \end{pmatrix}, \quad \overrightarrow{QR} = \begin{pmatrix} r-1 \\ 1-q \\ 1 \end{pmatrix}, \quad \overrightarrow{QS} = \begin{pmatrix} -1 \\ s-q \\ 1 \end{pmatrix} \text{ and } \overrightarrow{RS} = \begin{pmatrix} -r \\ s-1 \\ 0 \end{pmatrix} \quad \textbf{(*)}$$

(ii) Diagonal PR has eqn. $\mathbf{r} = \begin{pmatrix} p \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} r-p \\ 1 \\ 1 \end{pmatrix}$; diagonal QS has eqn. $\mathbf{r} = \begin{pmatrix} 1 \\ q \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ s-q \\ 1 \end{pmatrix}$

M1 Good attempt at both eqns.

Diagonals intersect iff

$p + \lambda(r-p) = 1 - \mu$, $\lambda = q + \mu(s-q)$, $\lambda = \mu$ **M1**

Setting $\mu = \lambda \Rightarrow p + \lambda(r-p) = 1 - \lambda$, $\lambda = q + \lambda(s-q)$ and equating for λ

M1

$\lambda = \dfrac{1-p}{r-p+1} = \dfrac{q}{1-s+q} \Rightarrow 1 - s + q - p + ps - pq = rq - pq + q$

$\Rightarrow 1 - s - p + ps = rq \Rightarrow (1-s)(1-p) = rq$ **A1** legitimately (**AG**) **4**

ALT.

Taking any 3 ('*independent*') vectors from **(*)** and showing them linearly dependent (consistently)

(ii)(a) Then $\overrightarrow{PQ} = \overrightarrow{SR}$ iff $1 - p = r$ **and** $q = 1 - s$

i.e. iff $p + r = 1$ **and** $q + s = 1$ **B1**

Centroid G has $\mathbf{g} = \left(\frac{1}{4}(p+r+1),\ \frac{1}{4}(q+s+1),\ \frac{1}{2}\right)$ **B1**

while the centre of the unit cube is at $\left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right)$

These are the same point iff $p + r = 1$ **and** $q + s = 1$ **B1**

Final mark *not* to be awarded unless a correct "iff" argument has been made **3**

(ii) (b) We have $p + r = 1$ **and** $q + s = 1$ as before

and $\sqrt{(1-p)^2 + q^2} = \sqrt{p^2 + s^2 + 1}$ using $\overrightarrow{PS}$ from **(*)** **B1**

$\Rightarrow 1 - 2p + p^2 + q^2 = p^2 + s^2 + 1$ **M1** solving for p

$\Rightarrow p = \dfrac{q^2 - s^2}{2} = \dfrac{(q-s)(q+s)}{2} = \dfrac{q-s}{2}$ using $q + s = 1$

Then $q + s = 1$ and $q - s = 2p \Rightarrow q = \frac{1}{2} + p$, $r = 1 - p$, $s = \frac{1}{2} - p$

All found in terms of p **M1 A1** **4**

$\overrightarrow{PQ} = \begin{pmatrix} 1-p \\ \frac{1}{2}+p \\ 0 \end{pmatrix}$, $\overrightarrow{PR} = \begin{pmatrix} 1-2p \\ 1 \\ 1 \end{pmatrix}$, $\overrightarrow{RQ} = \begin{pmatrix} p \\ p-\frac{1}{2} \\ -1 \end{pmatrix}$ **B1** must all be in terms of p (**ft**)

so $PQ^2 = 2p^2 - p + \frac{5}{4}$, $PR^2 = 4p^2 - 4p + 3$

and $RQ^2 = 2p^2 - p + \frac{5}{4}$ **M1** three lengths attempted

Then by the Cosine Rule,

$$\cos PQR = \frac{PQ^2 + RQ^2 - PR^2}{2.PQ.RQ} = \frac{2p - \frac{1}{2}}{2\left(2p^2 - p + \frac{5}{4}\right)}$$ **M1** (rearranged into cos = … form)

$$= \frac{4p-1}{5-4p+8p^2}$$ **A1** (**AG**) legitimately obtained **4**

ALT. $\overrightarrow{RQ} = \begin{pmatrix} 1-r \\ q-1 \\ -1 \end{pmatrix} = \begin{pmatrix} p \\ p-\frac{1}{2} \\ -1 \end{pmatrix}$ B1 must be all in terms of p (possibly later on)

$$\cos PQR = \frac{\overrightarrow{PQ} \bullet \overrightarrow{RQ}}{\left|\overrightarrow{PQ}\right| \left|\overrightarrow{RQ}\right|} = \frac{(1-p)p + \left(p+\frac{1}{2}\right)\left(p-\frac{1}{2}\right) + 0}{\sqrt{(1-p)^2 + \left(p+\frac{1}{2}\right)^2}\sqrt{p^2 + \left(p-\frac{1}{2}\right)^2 + 1}}$$

M2 use of the scalar product (correct vectors)

$$= \frac{p - p^2 + p^2 - \frac{1}{4}}{\sqrt{(1-p)^2 + \left(p+\frac{1}{2}\right)^2}\sqrt{p^2 + \left(p-\frac{1}{2}\right)^2 + 1}} = \frac{p - \frac{1}{4}}{\sqrt{\frac{5}{4} - p + 2p^2}\sqrt{\frac{5}{4} - p + 2p^2}}$$

$$= \frac{4p-1}{5-4p+8p^2}$$ A1 legitimately (**AG**) **4**

For a square, adjacent sides perpr. $\Rightarrow p = \frac{1}{4}, q = \frac{3}{4}, r = \frac{3}{4}, s = \frac{1}{4}$ **B1**

Side-length is $\left|\overrightarrow{PQ}\right| = \sqrt{\frac{5}{4} - p + 2p^2} = \sqrt{\frac{5}{4} - \frac{1}{4} + \frac{2}{16}} = \frac{3}{2\sqrt{2}}$ **B1**

$\frac{3}{2\sqrt{2}} > \frac{21}{20} \Leftarrow \frac{1}{\sqrt{2}} > \frac{7}{10} \Leftarrow 10 > 7\sqrt{2} \Leftarrow 100 > 98$ **B1** or equivalent

(penalise incorrect direction of the logic) **3**

ALT. (final two marks)

Side-length is $\sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^2}$ B1 $= \sqrt{\frac{9}{8}} = \sqrt{\frac{450}{400}} > \sqrt{\frac{441}{400}} = \frac{21}{20}$ B1

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6 (i) $9x^2 - 12x\cos\theta + 4 \equiv (3x - 2\cos\theta)^2 + 4 - 4\cos^2\theta$ **M1**

$\geq 4\sin^2\theta$ with equality when $x = \frac{2}{3}\cos\theta$ **A1 B1** **3**

(the value of x giving the minimum may appear later on)

$12x^2\sin\theta - 9x^4 \equiv 4\sin^2\theta - (3x^2 - 2\sin\theta)^2$ **M1**

$\leq 4\sin^2\theta$ with equality when $x^2 = \frac{2}{3}\sin\theta$ **A1 B1**

(the value of x giving the maximum may appear later on) **3**

ALT. $y = 12x^2\sin\theta - 9x^4 \Rightarrow \dfrac{dy}{dx} = 24x\sin\theta - 36x^3$ M1

$\dfrac{dy}{dx} = 0$ when $x^2 = \frac{2}{3}\sin\theta$, $y = 4\sin^2\theta$ B1 both

(ignore consideration of $x = 0$; this clearly does not give a max.)

$\dfrac{d^2y}{dx^2} = 24\sin\theta - 108x^2 = -48\sin\theta < 0$ for $0 < \theta < \pi \Rightarrow$ maximum

A1 must justify MAX. if using calculus

$9x^4 + (9 - 12\sin\theta)\,x^2 - 12x\cos\theta + 4 = 0$

$\Leftrightarrow 9x^2 - 12x\cos\theta + 4 = 12x^2\sin\theta - 9x^4$ **B1**

These two functions meet only at $4\sin^2\theta$ when $x^2 = \frac{4}{9}\cos^2\theta = \frac{2}{3}\sin\theta$

E1 explained

$\frac{4}{9}(1 - s^2) = \frac{2}{3}s \Rightarrow 0 = 2s^2 + 3s - 2 = (2s - 1)(s + 2)$ **M1** creating and solving a quadratic

$\Rightarrow \sin\theta = \frac{1}{2}$, $x^2 = \frac{1}{3}$

$\Rightarrow (x, \theta) = \left(\pm\dfrac{1}{\sqrt{3}}, \dfrac{\pi}{6}\right), \left(\pm\dfrac{1}{\sqrt{3}}, \dfrac{5\pi}{6}\right)$ **A1** at least two correct solutions

Checking for extraneous solutions, we find that only

$\left(\dfrac{1}{\sqrt{3}}, \dfrac{\pi}{6}\right)$ and $\left(-\dfrac{1}{\sqrt{3}}, \dfrac{5\pi}{6}\right)$ are valid solutions **B1** **5**

(ii) Vertical Asymptote $x = \theta$ **B1** stated or shown on graph

$y = \dfrac{x(x-\theta) + \theta(x-\theta) + \theta^2}{x - \theta} = x + \theta + \dfrac{\theta^2}{x - \theta}$

$\Rightarrow$ Oblique Asymptote $y = x + \theta$ **B1** stated or shown on graph

(NB OAs aren't on-syllabus so allow $y \to \pm\infty$ as $x \to \pm\infty$)

$\dfrac{dy}{dx} = 1 - \dfrac{\theta^2}{(x-\theta)^2} = 0$ when … **M1** method for finding TPs

$(x - \theta)^2 = \theta^2 \Rightarrow x = 0, y = 0$ or $x = 2\theta, y = 4\theta$ **A1** stated or shown on graph

From graph, $\dfrac{x^2}{x - \theta} \leq 0$ or $\dfrac{x^2}{x - \theta} \geq 4\theta$ **B1** graph must be correct **5**

Since $4\theta > 0$, we have $\dfrac{x^2}{4\theta(x-\theta)} \le 0$ or $\dfrac{x^2}{4\theta(x-\theta)} \ge 1$

so we have $\dfrac{\sin^2\theta\cos^2 x}{1+\cos^2\theta\sin^2 x} \le 0$ or $\dfrac{\sin^2\theta\cos^2 x}{1+\cos^2\theta\sin^2 x} \ge 1$

However, it is clear that $(0 \le) \dfrac{\sin^2\theta\cos^2 x}{1+\cos^2\theta\sin^2 x} \le 1$ **B1**

(since numerator ≤ 1 and denominator ≥ 1)

The 0 case occurs if and only if $x = 0$ (on LHS) but, since $\sin\theta \ne 0$, the RHS is then non-zero (as $\cos 0 = 1$)

$\dfrac{\sin^2\theta\cos^2 x}{1+\cos^2\theta\sin^2 x} = 1$ $\Leftrightarrow$ both numerator & denominator are 1 **M1**

$\Leftrightarrow$ $\sin^2\theta = 1$ **and** $\cos^2 x = 1$ **M1**

$\Leftrightarrow$ $\theta = \dfrac{\pi}{2}$, $x = \pi$ **A1** **4**

ALT. For the 1 case, we want $\sin^2\theta\cos^2 x = 1 + \cos^2\theta\sin^2 x$ when $x = 2\theta$

(from previous bit) B1

Thus $\sin^2\theta\cos^2 2\theta - \cos^2\theta\sin^2 2\theta = 1$

$\Rightarrow$ $(\sin\theta\ \cos 2\theta - \cos\theta\sin 2\theta)\,(\sin\theta\ \cos 2\theta + \cos\theta\sin 2\theta) = 1$

M1

$\Rightarrow$ $(-\sin\theta)\,(\sin 3\theta) = 1$ **or** $(4\sin^2\theta + 1)(\sin^2\theta - 1) = 0$

This can only occur when either $\sin\theta = 1$ and $\sin 3\theta = -1$

or $\sin\theta = -1$ and $\sin 3\theta = 1$ M1

Since $0 < \theta < \pi$, this is only satisfied when $\theta = \dfrac{\pi}{2}$, $x = \pi$ A1

7 **(i) Step 1:** If a is not divisible by 3 then it is either one more than, or one less than, a multiple of 3.

For $a = 3k \pm 1$, $a^2 = 9k^2 \pm 6k + 1$

$= 3(3k^2 \pm 2k) + 1$ — **B1** (both shown 1 more than a multiple of 3) — **1**

Step 3: $\left(\sqrt{2}+\sqrt{3}\right)^2 = 5 + 2\sqrt{6}$ — **B1**

$\left(\sqrt{2}+\sqrt{3}\right)^4 = 49 + 20\sqrt{6}$ — **M1** and relating back to a, b

$\left(\dfrac{a}{b}\right)^4 = 10\left(\dfrac{a}{b}\right)^2 - 1$ — **A1**

$\times$ by b^4 and rearranging gives $a^4 + b^4 = 10a^2b^2$

A1 legitimately (**AG**) — **4**

ALT. $a = \left(\sqrt{2}+\sqrt{3}\right)b \;\Rightarrow\; a^2 = \left(5+2\sqrt{6}\right)b^2$ — B1

$\Rightarrow\; a^2 + b^2 = \left(6+2\sqrt{6}\right)b^2$ — M1 adding b^2 to both sides

$= 2\sqrt{3}\left(\sqrt{3}+\sqrt{2}\right)b^2 = 2\sqrt{3}ab$ — A1

$\Rightarrow\; \left(a^2+b^2\right)^2 = 12a^2b^2 \;\Rightarrow\; a^4 + b^4 = 10a^2b^2$ — A1 legitimately (**AG**)

Step 4: If $a = 3k$ then $b^4 = 90k^2b^2 - 81k^4 = 3(30k^2b^2 - 27k^4)$

Explanation that $3 \mid \text{RHS} \Rightarrow 3 \mid \text{LHS} \Rightarrow 3 \mid b$

E1 must be thorough — **1**

Step 5: Since hcf(a, b) = 1, a can't be a multiple of 3 (from previous working)

B1

So both a^2 and $a^4 \equiv 1 \pmod 3$ — **M1** any suitable wording

giving $1 + b^4 \equiv b^2 \pmod 3$ — **A1**

Each case $b^2 \equiv 0$, $b^2 \equiv 1$ gives $\Rightarrow\Leftarrow$ — **E1** carefully explained — **4**

(ii) If a is not a multiple of 5, it is $5k \pm 1$ or $5k \pm 2$ — **B1**

Squaring gives $a^2 \equiv \pm 1$ (and $a^4 \equiv 1$) — **B1**

$\left(\sqrt{6}+\sqrt{7}\right)^2 = 13 + 2\sqrt{42}$

and $\left(\sqrt{6}+\sqrt{7}\right)^4 = 337 + 52\sqrt{42}$ — **M1** and relating back to a, b

so that $\left(\dfrac{a}{b}\right)^4 = 26\left(\dfrac{a}{b}\right)^2 - 1$ — **A1**

$\times$ by b^4 and rearranging gives $a^4 + b^4 = 26a^2b^2$ — **A1** legitimately (**AG**)

Now if $a = 5k$ then $b^4 = 650k^2b^2 - 625k^4 = 5(130k^2b^2 - 125k^4)$

so if a is a multiple of 5 then b is also — **E1**

Since a, b co-prime, this doesn't happen

so $a^4 + b^4 = 26a^2b^2$ becomes — **M1** considering this mod 5

$1 + b^4 \equiv \pm b^2$ — **A1**

Each case $b^2 \equiv 0$, $b^2 \equiv \pm 1$ gives $\Rightarrow\Leftarrow$ — **E1** carefully explained/demonstrated — **9**

Cannot work with divisibility by 3 in this case since $26 \equiv 2 \pmod 3$ and $a^4 + b^4 = 26a^2b^2$

could be from $1 + 1 \equiv 2.1.1$ (e.g.) **E1** must demonstrate the failing case explic

8	**(i)** Set $u = 2t$, $\mathrm{d}u = 2\mathrm{d}t$ in $\mathrm{f}(x) = \int_1^x \sqrt{\frac{t-1}{t+1}}\,\mathrm{d}t$	**M1** choice of substitution	
	$t = 1,\ u = 2$ and $t = \frac{1}{2}x,\ u = x$	**A1** limits correctly sorted	
	Then $\mathrm{f}\left(\frac{1}{2}x\right) = \int_2^x \sqrt{\frac{\frac{u}{2}-1}{\frac{u}{2}+1}}\,.\frac{1}{2}\mathrm{d}u = \frac{1}{2}\int_2^x \sqrt{\frac{u-2}{u+2}}\,\mathrm{d}u$	**M1 A1** full substitution attempted; correct	
	and $\int_2^x \sqrt{\frac{u-2}{u+2}}\,\mathrm{d}u = 2\,\mathrm{f}\left(\frac{1}{2}x\right)$	**A1** legitimately (**AG**)	
	A 'backwards' verification approach equally ok		**5**
	(ii) Set $u = v + 2$, $\mathrm{d}u = \mathrm{d}v$	**M1** choice of substitution, using **(i)**	
	$u = 2,\ v = 0$ and $u = x + 2,\ v = x$	**A1** limits correctly sorted	
	Then $2\,\mathrm{f}\left(\frac{x+2}{2}\right) = \int_0^x \sqrt{\frac{v}{v+4}}\,\mathrm{d}v$	**A1**	**3**

ALT. (from the beginning)

Set $u + 2 = 2t$, $\mathrm{d}u = 2\,\mathrm{d}t$ — M1 choice of substitution

$t = 1,\ u = 0$ and $t = \frac{1}{2}x + 1,\ u = x$

Then $\mathrm{f}\left(\frac{1}{2}x+1\right) = \int_1^{\frac{1}{2}x+1} \sqrt{\frac{t-1}{t+1}}\,\mathrm{d}t = \int_0^x \sqrt{\frac{\frac{u+2}{2}-1}{\frac{u+2}{2}+1}}\,.\frac{1}{2}\mathrm{d}u$ — M1 full substitution

$= \frac{1}{2}\int_0^x \sqrt{\frac{u}{u+4}}\,\mathrm{d}u$

and $2\,\mathrm{f}\left(\frac{1}{2}x+1\right) = \int_0^x \sqrt{\frac{u}{u+4}}\,\mathrm{d}u$ — A1

(iii)	Set $u = at + b$, $\mathrm{d}u = a\,\mathrm{d}t$	**M1** choice of substitution	
	$t = 1,\ u = 5$ and $t = \frac{x-b}{a},\ u = x$	**A1** limits correctly sorted	
	Then $\mathrm{f}\left(\frac{x-b}{a}\right) = \int_1^{\frac{x-b}{a}} \sqrt{\frac{t-1}{t+1}}\,\mathrm{d}t = \int_5^x \sqrt{\frac{\frac{u-b}{a}-1}{\frac{u-b}{a}+1}}\,\frac{1}{a}\mathrm{d}u$	**M1** full substitution	
	$= \frac{1}{a}\int_5^x \sqrt{\frac{u-(a+b)}{u+(a-b)}}\,\mathrm{d}u$	**A1** correct	
	We need $a + b = 5$ and $a - b = 1 \Rightarrow a = 3, b = 2$	**M1** method for determining a, b	
	giving $3\mathrm{f}\left(\frac{x-2}{3}\right) = \int_5^x \sqrt{\frac{u-5}{u+1}}\,\mathrm{d}u$	**A1**	
	Might also be done using two substitutions (split marks **3 + 3** if fully correct)		**6**
(iv)	Set $y = u^2$, $\mathrm{d}y = 2u\,\mathrm{d}u$	**M1** choice of substitution	
	$u = 1,\ y = 1$ and $u = 2,\ y = 4$	**A1** limits correctly sorted	

Then $\int_1^2 \sqrt{\frac{u^2}{u^2+4}}\, u\, du = \int_1^4 \sqrt{\frac{y}{y+4}}\, \frac{1}{2}\, dy$ **M1** full substitution

$= \frac{1}{2}\int_0^4 \sqrt{\frac{y}{y+4}}\, dy - \frac{1}{2}\int_0^1 \sqrt{\frac{y}{y+4}}\, dy$ **M1** dealing with lower limit

$= f\left(\frac{4+2}{2}\right) - f\left(\frac{1+2}{2}\right)$ using **(ii)** **M1** use of **(ii)**

$= f(3) - f(\frac{3}{2})$ **A1** **6**

For those interested, $f(x) = \sqrt{x^2-1} - 2\sinh^{-1}\sqrt{\frac{x-1}{2}}$ or equivalent involving log forms

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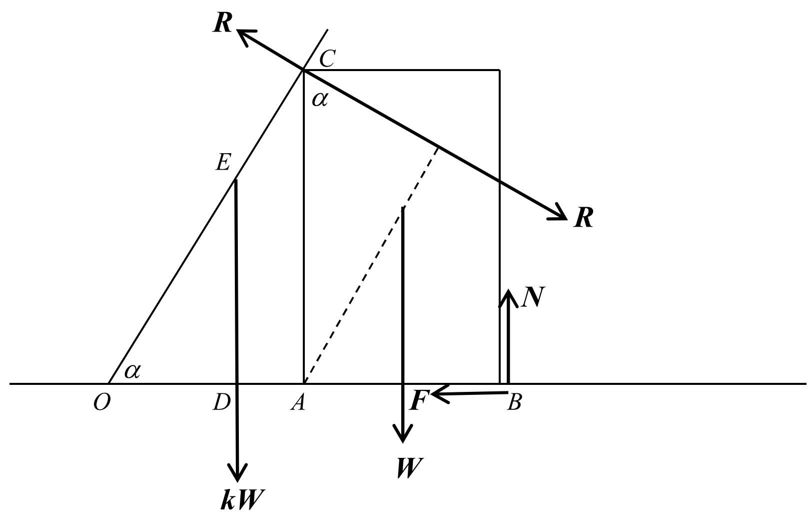
9 Diagram at the moment of toppling:-

Note: There could also be ↑ and → components of the contact force at O, but these can be ignored



$OA = AB = b$ and $AC = h \Rightarrow OC = \sqrt{b^2 + h^2}$

$DE = \lambda h$ and $OD = \lambda b$

(i) $\circlearrowleft O$ for ladder: $kW.\lambda b = R\sqrt{b^2+h^2}$ **M1 A1**

$\Rightarrow R = k\lambda W \dfrac{b}{\sqrt{b^2+h^2}} = k\lambda W \cos\alpha$ **M1 A1** legitimately (**AG**) **4**

(ii) Resolve ↑ for box: $N = W + R\cos\alpha$ **M1 A1**

$\circlearrowleft A$ for box: $W.\tfrac{1}{2}b + R.h\sin\alpha = N.b$ **M1 A1**

$\Rightarrow \tfrac{1}{2}b\,W + hk\lambda W\cos\alpha\sin\alpha = b(W + k\lambda W\cos^2\alpha)$ **M1 M1** substituting for R, N

$\Rightarrow \tfrac{1}{2}b + hk\lambda\cos\alpha\sin\alpha = b + bk\lambda\cos^2\alpha$

$(\times 2) \Rightarrow b\tan\alpha \,.\, 2k\lambda\cos\alpha\sin\alpha = b + 2bk\lambda\cos^2\alpha$ **M1** substituting $h = b\tan\alpha$

$(\div b) \Rightarrow 2k\lambda\sin^2\alpha = 1 + 2k\lambda\cos^2\alpha$

$\Rightarrow 0 = 1 + 2k\lambda\cos 2\alpha$ **A1** (since $c^2 - s^2 = \cos 2\alpha$) legitimately (**AG**) **8**

(iii) Resolve → for box: $F = R\sin\alpha$ **B1**

Friction Law: $F \le \mu N$ **B1** *used*, not just stated (**B0** for $F = \mu N$)

$\Rightarrow \mu \ge \dfrac{R\sin\alpha}{W + R\cos\alpha}$ **M1** substituting for F and N

$\Rightarrow \mu \ge \dfrac{k\lambda W\cos\alpha\sin\alpha}{W + k\lambda W\cos^2\alpha}$ **M1** substituting for R

$\Rightarrow \mu \ge \dfrac{k\lambda 2\sin\alpha\cos\alpha}{2 + k\lambda 2\cos^2\alpha} = \dfrac{k\lambda\sin 2\alpha}{2 + k\lambda(1+\cos 2\alpha)}$ **M1** use of double-angle formulae

Using (ii)'s result, $1 = -2k\lambda\cos 2\alpha \Rightarrow 2 = -4k\lambda\cos 2\alpha$ **M1 A1** using (ii)'s result

$$\Rightarrow \qquad \mu \geq \frac{k\lambda \sin 2\alpha}{-4k\lambda\cos 2\alpha + k\lambda + k\lambda\cos 2\alpha}$$

$$(\div\, k\lambda) \Rightarrow \qquad \mu \geq \frac{\sin 2\alpha}{-4\cos 2\alpha + 1 + \cos 2\alpha} = \frac{\sin 2\alpha}{1 - 3\cos 2\alpha}$$ **A1** legitimately (**AG**) **8**

10	$x = ut\sin\alpha \qquad y = ut\cos\alpha - \frac{1}{2}gt^2$	**B1** both	
	Setting $t = \dfrac{x}{u\sin\alpha}$ and substituting into y formula	**M1**	
	$\Rightarrow y = x\cot\alpha - \frac{1}{2}g\dfrac{x^2}{u^2\sin^2\alpha}$		
	$= x\cot\alpha - \frac{1}{2}g\dfrac{x^2}{u^2}\left(1+\cot^2\alpha\right)$	**M1** use of $\operatorname{cosec}^2\alpha = 1 + \cot^2\alpha$	
	Setting $x = h\tan\beta$ and $y = h$	**M1**	
	$\Rightarrow h = ch\tan\beta - \dfrac{gh^2}{2u^2}\tan^2\beta\left(1+c^2\right) \Rightarrow$ (since $h \neq 0$) $1 = c\tan\beta - \dfrac{gh}{2u^2}\tan^2\beta\left(1+c^2\right)$		
	$\times\ k = \dfrac{2u^2}{gh} \Rightarrow k = ck\tan\beta - (1+c^2)\tan^2\beta$	**M1** use of k	
	$\div \tan^2\beta \quad \Rightarrow k\cot^2\beta = ck\cot\beta - 1 - c^2$		
	$\Rightarrow c^2 - ck\cot\beta + 1 + k\cot^2\beta = 0$	**A1** legitimately (**AG**)	**6**
(i)	Considering this quadratic in c:		
	sum of roots: $\quad \cot\alpha_1 + \cot\alpha_2 = k\cot\beta$	**B1** must be clearly shown (**AG**)	
	product of roots: $\quad \cot\alpha_1\cot\alpha_2 = 1 + k\cot^2\beta$	**B1** stated or used somewhere	
	$\cot(\alpha_1+\alpha_2) = \dfrac{1}{\tan(\alpha_1+\alpha_2)} = \dfrac{1-\tan\alpha_1\tan\alpha_2}{\tan\alpha_1+\tan\alpha_2}$	**M1** tan/cot$(A+B)$ result	
	$= \dfrac{\cot\alpha_1\cot\alpha_2 - 1}{\cot\alpha_1+\cot\alpha_2}$	**M1** everything in terms of cots	
	$= \dfrac{1+k\cot^2\beta - 1}{k\cot\beta} = \cot\beta$	**A1**	
	and it follows that $\alpha_1 + \alpha_2 = \beta$ ($\because \beta, \alpha_1, \alpha_2$ all acute)	**E1** (**AG**) must be justified	**6**
	Still considering the quadratic in c:		
	For real c, discriminant $\Delta = (k\cot\beta)^2 - 4(1 + k\cot^2\beta) \geq 0$	**M1** considering discriminant	
	$\Rightarrow (k^2-4k)\cot^2\beta \geq 4 \Rightarrow k^2 - 4k \geq 4\tan^2\beta$		
	$\Rightarrow (k-2)^2 \geq 4\tan^2\beta + 4 = 4\sec^2\beta$	**M1** completing the square and trig. identity used	
	$\Rightarrow \quad k \geq 2(1+\sec\beta) \qquad$ (… ignore $k \leq$ –ve thing)	**A1** legitimately (**AG**)	**3**
(ii)	$\dot{y} = u\cos\alpha - gt \Rightarrow t = \dfrac{u\cos\alpha}{g}$ at max. height	**M1** stated or used in y-formula	
	$\Rightarrow H = \dfrac{u^2\cos^2\alpha}{2g}$	**A1** (give M1 A1 if result correctly quoted)	
	$h \leq H \Rightarrow 2gh \leq u^2\cos^2\alpha$	**M1** comparing h with H	
	$\Rightarrow 2 \times \dfrac{2u^2}{k} \leq u^2\cos^2\alpha$	**M1** substituting for k	
	$\Rightarrow k \geq 4\sec^2\alpha$	**A1** (**AG**) legitimately obtained	**5**

11 **(i)**	P(HH) = p^2 P(TT) = q^2 P(TH or HT) = $2pq$	**B1** seen at any stage	
	P(first n–1 rounds indecisive) = $(2pq)^{n-1}$		
	$\Rightarrow$ Decision at round n = $(2pq)^{n-1}$ × P(HH or TT)	**M1**	
	= $(2pq)^{n-1}(p^2+q^2)$	**A1** legitimately (**AG**)	**3**
	Let d = P(decision on or before n^{th} round)		
	= 1 – P(decision after n^{th} round)	**M1** with working	
	$= 1 - \left\{(2pq)^n(p^2+q^2) + (2pq)^{n+1}(p^2+q^2) + (2pq)^{n+2}(p^2+q^2) + \ldots\right\}$		
	$= 1 - (2pq)^n(p^2+q^2)\left\{1 + (2pq) + (2pq)^2 + \ldots\right\}$		
	$= 1 - (2pq)^n(p^2+q^2) \times \dfrac{1}{1-2pq}$	**M1** use of S_∞(GP)	
	since $p^2+q^2 = (p+q)^2 - 2pq = 1 - 2pq$		
	$= 1-(2pq)^n$	**A1**	
	Now $\sqrt{pq} \le \frac{1}{2}(p+q) = \frac{1}{2}$ by the AM-GM inequality	**M1 A1** method; correct	
	or via $\left(\sqrt{p}-\sqrt{q}\right)^2 \ge 0 \Rightarrow p+q-2\sqrt{pq} \ge 0$ etc.		
	or via $pq = p(1-p) \le \frac{1}{4}$ by calculus/completing the square	**E1** inequality concluded	
	and $d = 1-(2pq)^n = 1-2^n\left(\sqrt{pq}\right)^{2n} \ge 1-2^n\left(\tfrac{1}{2}\right)^{2n} = 1-\dfrac{1}{2^n}$	**A1** legitimately (**AG**)	**7**
(ii)	P(decision at 1$^{\text{st}}$ round) = p^3+q^3 or $1-3pq$	**B1**	
	P(decision at 2$^{\text{nd}}$ round) = $3p^2q.p^2 + 3q^2p.q^2$	**M1** Good attempt at two cases	
	So overall prob. is $P = p^3+q^3+3p^4q+3pq^4$	**A1**	**3**
	$P = p^3 + (1-p)^3 + 3(p^4-p^5) + 3p(1-p)^4$	**M1** a polynomial in p only	
	$= 1 - 9p^2 + 18p^3 - 9p^4$	**A1**	
	$\dfrac{\mathrm{d}P}{\mathrm{d}p} = -18p + 54p^2 - 36p^3$	**M1**	
	$= -18p(2p-1)(p-1)$	**M1** and set to zero	
	giving $p = 0, \frac{1}{2}, 1$	**A1**	
	Since P is a positive cubic, 0 and 1 give maxima, while $\frac{1}{2}$ gives a (local) minimum	**E1** justification	
	So, on [0, 1], $p = \frac{1}{2}$ and $P_{\min} = \frac{7}{16}$	**A1**	**7**

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